Computing 43, 401–410 (1990)

Computing

Yet Another Application of a Binomial Recurrence. Order Statistics

Wojciech Szpankowski* and **Vernon Rego***, West Lafayette, Ind.

Received May 23, 1989

Abstract — Zusammenfassung

Yet Another Application of a Binomial Recurrence. Order Statistics. We investigate the moments of the maximum of a set of i.i.d geometric random variables. Computationally, the exact formula for the moments (which does not seem to be available in the literature) is inhibited by the presence of an alternating sum. A recursive expression for the moments is shown to be superior. However, the recursion can be both computationally intensive as well as subject to large round-off error when the set of random variables is large, due to the presence of factorial terms. To get around this difficulty we develop accurate asymptotic expressions for the moments and verify our results numerically.

Key words: geometric distribution, order statistics, binomial recurrence, asymptotic approximation, program unification, concurrency enhancement.

Eine weitere Anwendung binomischer Rekurrenz. Orderstatistik. Wir untersuchen die Momente des Maximums einer Menge von unabhängig identisch verteilten geometrischen Zufallsvariablen. Numerisch ist die Verwendung der exakten Formel für die Momente (die überdies in der Literatur nicht erscheint) wegen des Vorhandenseins einer alternierenden Summe nicht ratsam. Ein rekursiver Ausdruck für die Momente ist besser geeignet. Jedoch kann die Rekursion wegen des Auftretens von faktoriellen Ausdrücken sowohl viel Rechenaufwand erfordern als auch große Rundungsfehler verursachen, wenn die Menge der Zufallsvariablen groß ist. Zur Überwindung dieser Schwierigkeiten entwickeln wir genaue asymptotische Formeln für die Momente und verifizieren unsere Ergebnisse numerisch.

1. Introduction

In this paper, we present some results concerning the maximum order statistic $\mathbf{M}_n$ of n independent and identically distributed geometric random variables. We first develop a recurrence, and then an exact formula for the mean and variance of this order statistic. To the best of our knowledge, these results appear to be new (see, for example [2, 3]). The exact formula for the first two moments of this order statistic is computationally unsatisfactory, since it involves an alternating sum. The recurrence is computationally sound, but due to the presence of a factorial term, is not to be recommended for very large values of n. In order to get around this barrier, we develop asymptotic forms for all the moments of this order statistic. In particular, we present extremely accurate asymptotic results for the mean and variance of $\mathbf{M}_n$.

* This research was partially supported by NSF under grants NCR-8702115 and CCR-8900305.

The motivation for this study came from a model of a program unification technique in concurrency enhancement methods [6]. The unification technique essentially combines n copies of a poorly vectorizable piece of code into a single, vectorizable piece of code. The average speedup of the single piece of code, when run on a vector/concurrent machine such as the Alliant FX/80 can be shown to be significant [6]. In particular, consider the case of programs that are composed of strings of iterative blocks. A simple nondeterministic model of program behaviour suggests that each program block goes through a random number of iterations, where this random number is geometrically distributed. Executing a unified iterative program block, under the geometric model, is like watching n simultaneously initiated, independent, and identical geometric random variables expire, with the execution time of the unified block taken to be the largest of these times. If the original piece of code contains many types of blocks, the unified piece of code will also contain this many block types, but the limited number of processors on the target machine dictates that efficient scheduling policies are required for efficient execution of the unified code. Studying average speedup for large values of n for most models, in particular the geometric, takes us into the study of asymptotics.

The simplest model of the unified execution scheme is the following. Consider a set of n balls placed in an urn at time $k = 0$. At each discrete time step k, $k \geq 1$, we are required to pick up all the balls in the urn and toss them into the air. For each ball that is tossed, and for each value of k, there is nonzero probability p that the ball will fall out of the urn, for $k \geq 1$. Balls that fall out of the urn are ignored. We are interested in the number of tosses that is required to empty out the urn. The number of tosses required for any one ball (say that i^{th}) to fall out of the urn is given by a geometric random variable $\mathbf{X}_i$. We assume the balls do not influence one another when tossed, so that the random variables $\mathbf{X}_i$, $1 \leq i \leq n$, are i.i.d. It follows that the number of tosses required to empty the urn is precisely the maximum, $\mathbf{M}_n = \max\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$, of this set of random variables. The original model in [6] investigates the situation where the number of urns is variable, and different rules may be used when choosing urns from which throws are to be made. In this paper we restrict our attention to $\mathbf{M}_n$ and accurate asymptotics for its moments.

In Section 2.1 we present the problem formally and prove a proposition concerning the exact solution and asymptotic approximations for the first two moments of $\mathbf{M}_n$. In the next two subsections we prove the main results, that is, in section 2.2 we develop a general solution to some binomial recurrences, and in section 2.3 we present an asymptotic solution for some alternating sums, and apply it to our problem. Finally, in section 3 we briefly present some computational results demonstrating the accuracy of the asymptotics, and conclude the paper.

2. Main Results

In this section, we present our main results. We start with a short, but formal, description of the problem, and we formulate our main results in the form of a proposition. In the next two subsections we provide a proof of the proposition.

2.1. *Problem Statement*

Let $\mathbf{X}_i$, $i = 1, 2, \ldots, n$ be a set of i.i.d random variables distributed according to the geometric distribution with parameter p. That is, for every $i = 1, 2, \ldots, n$,

$$Pr\{\mathbf{X}_i = k\} = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots, \tag{2.1a}$$

$$E\mathbf{X}_i = p^{-1}, \qquad E\mathbf{X}_i^2 = (2-p)p^{-2} \tag{2.1b}$$

We shall investigate, in particular, the first two moments of the maximum of $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$. Let

$$\mathbf{M}_n = \max\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\} \tag{2.2a}$$

and define

$$M_n = E\mathbf{M}_n, \qquad M_n^{(2)} = E\mathbf{M}_n^2 \tag{2.2b}$$

Then, M_n and $M_n^{(2)}$, as we shall show below, satisfy the following recurrences

$$M_0 = 0 \qquad M_1 = p^{-1} \tag{2.3a}$$

$$M_n = 1 + \sum_{k=0}^{n} \binom{n}{k} (1-p)^k p^{n-k} M_k \qquad n \geq 2 \tag{2.3b}$$

and

$$M_0^{(2)} = 0 \qquad M_1^{(2)} = (2-p)p^{-2} \tag{2.4a}$$

$$M_n^{(2)} = -1 + 2M_n + \sum_{k=0}^{n} \binom{n}{k} (1-p)^k p^{n-k} M_k^{(2)} \tag{2.4b}$$

To derive (2.3) and (2.4), we adopt a slight variation of the urn scheme discussed in the introduction. The intention is obtain an intuitive view of $\mathbf{M}_n$. Let us assume that n balls are put into n distinct urns, with one ball in each urn. The action of all urns is synchronized in the sense that at the beginning of a slot time (e.g., every second) each nonempty urn attempts to get rid of its ball by tossing it into the air. For each ball tossed, the ball falls out of its urn with probability p, and falls back into its urn with probability $q = 1 - p$. Each urn acts independently of the other urns. Note that $\mathbf{M}_n$ defined in (2.2a), is equal to the number of slots needed to empty all urns. After the first slot we have k nonempty urns, with probability $\binom{n}{k}(1-p)^k p^{n-k}$, and the time to empty these is equal to $\mathbf{M}_k$. This suggests the following recurrence for the l-th moment $E\mathbf{M}_n^l$ of $\mathbf{M}_n$

$$E\mathbf{M}_n^l = \sum_{k=0}^{n} \binom{n}{k} (1-p)^k p^{n-k} E(1 + \mathbf{M}_k)^l \tag{2.5}$$

Solving (2.5) for $l = 1$ and 2, one finally obtains our recurrences (2.3) and (2.4).

Using recurrences (2.3) and (2.4), we prove our main results, which can be summarized as follows.

Proposition. (i) *The exact solution for the first moment M_n of $\mathbf{M}_n$ is*

$$M_n = -\sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{1-q^k} \tag{2.6}$$

where $q \stackrel{\text{def}}{=} 1-p$, and for the second moment we obtain

$$M_n^{(2)} = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{1-q^k} - 2\sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{(1-q^k)^2} \tag{2.7}$$

(ii) *For large n the following asymptotics hold for the average M_n of $\mathbf{M}_n$*

$$M_n = \frac{\log n}{\log Q} + \frac{\gamma}{\log Q} + \frac{1}{2} + P_0(\log_Q n) + O(n^{-1}) \tag{2.8}$$

and for the variance of $\mathbf{M}_n$,

$$\operatorname{var} \mathbf{M}_n = \frac{\pi^2}{6\log^2 Q} + \frac{1}{12} + F(\log_Q n) \tag{2.9}$$

where log *denotes the natural logarithm, $Q \stackrel{\text{def}}{=} q^{-1}$ and $\gamma = 0.577\ldots$ is the Euler constant. The fluctuating functions $P_r(x)$ and $F(x)$ are defined as follows, with $\chi_k \stackrel{\text{def}}{=} 2\pi ik/\log Q$, $k = 0, \pm 1, \ldots,$*

$$P_r(x) = \frac{1}{\log Q} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \Gamma(r+\chi_k)\exp[-2\pi ikx] \quad r = 0, \pm 1, \pm 2, \ldots, \tag{2.10}$$

and $F(x) = G(x) + 2\cdot P_1(x) - P_0^2(x)$ where

$$G(x) = \frac{1}{\log^2 Q} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} [\Gamma(-\chi_k)\cdot \log_Q n - \Gamma'(-\chi_k)] \exp(2\pi ikx) \tag{2.11}$$

and $\Gamma(x)$ is the gamma function [1].

(ii) *The k-th moment $E\mathbf{M}_n^k$ of $\mathbf{M}_n$ becomes for large n*

$$E\mathbf{M}_n^k = \log_Q^k n + O(\log^{k-1} n) \tag{2.12}$$

□

The rest of the paper is devoted to proving the proposition and demonstrating the accuracy of the asymptotic results. Before that, however, we offer some additional remarks concerning the fluctuating function $P_r(x)$ and the term $O(n^{-1})$ in (2.8) and (2.9).

Remarks

(i) The fluctuating function $P_r(x)$ was studied by Knuth [5] and others [4], [7], [9]. In particular, the following properties of the function can be easily established:

- $P_r(x)$ is periodic function of $\log_Q n$. Indeed, $P_r(\log_Q n \cdot Q) = P_r(\log n)$.
- The function is bounded. This is proved by using the following properties of the gamma function [1], [4]

$$|\Gamma(it)|^2 = \frac{\pi}{t\sinh \pi t}, \qquad \Gamma(z+1) = z\cdot\Gamma(z)$$

- For any fixed ξ we have $P_r(\log n - \xi) = P_r(\log n) + O(\xi^{-1})$ since $\log(n - \xi) = \log n + O(\xi^{-1})$

As a consequence of the second property we may estimate the function $P_r(x)$. In particular, Knuth [5, p. 612] computed the upper bound $\bar{P}_1(\log_Q n)$ for the function $P_1(x)$, and the table below summarizes the results.

Q	$P_1(\log_Q n)$
2	0.000000175
3	0.00004122
4	0.000296
5	0.00085
10	0.0063
100	0.068
1000	0.153
1000000	0.341

This table, as well as the above established properties, suggest that the periodic function $P_r(x)$ has very small amplitude, and can be safely ignored in practice, if one uses (2.8) and (2.9).

(ii) The term $O(n^{-1})$ can be eliminated from the asymptotic expansions (2.8) and (2.9), and the accuracy of these formulas can be reduced to $O(n^{-M})$ for arbitrary $M > 0$, as explained in [9]. In such a case, the term $\log n$ should be replaced by the harmonic number $H_n = \sum_{i=1}^{n} 1/n$ [9]. If one develops the harmonic number H_n into an asymptotic expansion, then $H_n = \log n + \gamma + 1/2n + O(n^{-2})$ [5], hence the term $O(n^{-1})$ comes out. Since the constant $1/n$ is small, the contribution of the term $O(n^{-1})$ to the error is very small for large n.

2.2. *Exact Solution*

In this subsection, we prove part (i) of the proposition. Note that both recurrences (2.3) and (2.4) fall into the following general recurrence: given x_0 and x_1, solve for $n \geq 2$

$$x_n = a_n + \sum_{k=0}^{n} \binom{n}{k} q^k p^{n-k} x_k \tag{2.13}$$

where $p + q = 1$, and a_n is any sequence of numbers which we further call an *additive term* of the recurrence (2.13). In our case $a_n = 1$ for (2.3) and $a_n = 1 + 2M_n$ for (2.4). This type of recurrence was extensively studied in [7, 8] (see also [5]). We quote some results from [7, 8] here.

Let us define a sequence $\hat{a}_n$, called binomial inverse relations [5], as

$$\hat{a}_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k \qquad a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \hat{a}_k \tag{2.14}$$

Then,

Lemma 1. (i) *The recurrence* (2.13) *possesses the following solution*

$$x_n = x_0 + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{\hat{a}_k + kA - a_0}{1 - q^k} \tag{2.15}$$

where $A = a_1 - x_1 p - x_0 q$.

(ii) *The inverse* $\hat{x}_n$ *of* x_n *satisfies*

$$\hat{x}_n = \frac{\hat{a}_n + nA - a_0}{1 - q^n} \qquad n \geq 1 \tag{2.16}$$

Proof. The details are given in [7]. Here we offer only sketch of the derivation for the completeness of the presentation. To prove (2.15), we multiply (2.13) by $\frac{z^n}{n!}$ and computing the exponential generation function $X(z) = \sum_{n=0}^{\infty} x_n \frac{z^n}{n!}$ we obtain the following functional equation

$$X(z) - X(qz)e^{(1-q)z} = A(z) - a_0 - Az \tag{2.17}$$

where $A(z)$ is the exponential generation function for a_n. Introducing $H(z)X(z)e^{-z}$ we transform (2.17) into

$$H(z) - H(qz) = A(z)e^{-z} - a_0 e^{-z} - Aze^{-z}$$

This functional equation is easy to solve by consecutive iterations. Noting, in addition, that $\hat{a}_n$ has generating function $\hat{A}(-z) = A(z)e^{-z}$ we prove, after some algebra, Eq. (2.15). The proof of (2.16) is immediate by comparing (2.15) with the definition (2.14). □

Using our Lemma 1, the solution of (2.3) is simple, since one computes $\hat{a}_n = \delta_{n,0}$, where $\delta_{n,0}$ is the Kronecker delta [5], and $A = 1 - M_1 p = 0$. Hence (2.6) follows. To solve (2.4) we need the inverse $\hat{M}_n$ of M_n. But, from (2.6) and (2.16) we find $\hat{M}_n = -(1 - q^n)^{-1}$ for $n \geq 1$, and then (2.7) follows from Lemma 1 and some simple algebra.

2.3. *Asymptotic approximation*

The exact solutions given in Proposition (i) are not attractive from a numerical viewpoint. In fact, to compute M_n and $M_n^{(2)}$ one would do well to use the recurrences (2.3) and (2.4) instead of the exact solutions (2.6) and (2.7). Nevertheless, the exact formulas allow us to obtain very sharp asymptotic approximations which become numerically important for large n due to factorial terms in the recurrences.

Note that formulas (2.6) and (2.7) fall into the general pattern

$$S_n = \sum_{k=1}^{n} (-1)^k \binom{n}{k} f_k \tag{2.18}$$

where f_k is any sequence. It turns out to be useful to plug into complex analysis to obtain asymptotics of (2.18) [5], [9]. Let us assume that f_k has an analytical continuation $f(z)$, and $f(z)$ does not grow too fast to infinity (for detailed conditions see [9]), that is, $f(k) = f_k$. In [9], we have proved

Lemma 2. *The alternating sum S_n as defined in* (2.18) *can be represented by a complex integral as follows*

$$S_n = \frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \Gamma(z) f(-z) n^{-z}\,dz + e_n \tag{2.19}$$

where $\Gamma(z)$ is the gamma function [1,4], *and e_n is the error function given by*

$$e_n = O(n^{-1})\int_{-1/2-i\infty}^{-1/2+i\infty} z\Gamma(z) f(-z) n^{-z}\,dz \tag{2.20}$$

Proof. Formula (2.19) follows from Cauchy's theorem [4], and some algebraic manipulation. □

To apply Lemma 2, by Cauchy's residue theorem, we find residues of the function under the integral *right* of the line $(-1/2 - i\infty, 1/2 + i\infty)$. Then

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k} f_k = -\sum_{k=-\infty}^{\infty} \text{res}\{\Gamma(z_k) f(-z_k) n^{-z_k}\} + e_n + O(n^{-M}) \tag{2.21}$$

for any $M > 0$, and the sum is taken over all poles z_k, $k = 0, \pm 1, \pm 2, \ldots$, of the function under the integral (2.19).

To illustrate Lemma 2, we apply it to the asymptotic analysis of M_n and $M_n^{(2)}$. From (2.6) and (2.19) one finds

$$M_n = -\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\Gamma(z)n^{-z}}{1-q^{-z}}\,dz + e_n = \sum_{k=-\infty}^{\infty} \text{res}\left\{\frac{\Gamma(z_k)n^{-z_k}}{1-q^{-z_k}}\right\} + e_n$$

But, the poles z_k are the roots χ_k of the denominator, that is,

$$\chi_k = \frac{2\pi i k}{\log Q} \qquad k = 0, \pm 1, \ldots. \tag{2.22}$$

It is well known that the leading factor in the asymptotics comes from $k = 0$. In fact, $\chi_0 = 0$ is a double pole, since zero is also a pole of the gamma function [1,4]. To compute the residue at $\chi_0 = 0$, we use the following Taylor expansions [1]

$$\Gamma(z) = z^{-1} - \gamma + O(z) \tag{2.23a}$$

$$n^{-z} = 1 - z\log n + O(z^2) \tag{2.23b}$$

$$\frac{1}{1-q^{-z}} = -\frac{z^{-1}}{\log q^{-1}} + \frac{1}{2} + O(z) \tag{2.23c}$$

Multiplying (2.23a) through (2.23c) and finding the coefficient at z^{-1} leads to the residue at $\chi_0 = 0$. That is,

$$\text{res}_{\chi_0}\left\{\frac{\Gamma(\chi_0)n^{-\chi_0}}{1-q^{-\chi_0}}\right\} = \frac{\log n}{\log Q} + \frac{\gamma}{\log Q} + \frac{1}{2}$$

where $Q = q^{-1}$. The other poles, that is, χ_k for $k \neq 0$, contribute to the periodic function $P_0(x)$ defined in (2.10) (see Remark (i)). The error function e_n contributes $O(n^{-1})$. Indeed, it is enough to apply the residue theorem, and to see that the integral in (2.20) is $O(1)$.

The evaluation of $M_n^{(2)}$ is a little more intricate. Noting that $M_n^{(2)} = -M_n + D_n$, where

$$D_n = -\sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{1}{(1-q^k)^2} \tag{2.24}$$

and the asymptotics for M_n have already been computed, we focus our attention on D_n. An application of Lemma 2 yields

$$D_n = -\frac{2}{\log Q} + P_1(n) + O(n^{-1}) \tag{2.25}$$

where

$$P_1(n) = \frac{1}{\log Q} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \Gamma(1 + 2\pi ik/\log Q) \exp[-2\pi ik \log_Q n]$$

Now, for D_n we immediately obtain from Lemma 2

$$D_n = \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\Gamma(z) n^{-z}}{(1-q^{-z})^2} dz \tag{2.26}$$

But, this time the pole $\chi_0 = 0$ is a triple pole of the function under the integral. This case was extensively studied in [8], and we show there that the following expansions must be used

$$\begin{aligned} \Gamma(z) &= z^{-1} - \gamma + \left(\frac{\pi^2}{12} + \frac{\gamma^2}{2}\right) z + O(z^2) \\ n^{-z} &= 1 - z\log n + \frac{z^2}{2}\log^2 n + O(z^3) \\ (1-q^{-z})^2 &= z^2(b_0 + b_1 z + b_2 z^2 + O(z^3)) \end{aligned} \tag{2.27}$$

where $b_0 = h_1^2$, $b_1 = h_1 h_2$ and $b_2 = \frac{1}{4}h_2^2 + \frac{1}{3}h_1 h_3$ and $h_n = (-1)^n \log^n q$. In [8] an algorithm is given to compute the residue in such a case (note that the last equation in (2.27) shows the Taylor expansion of $(1-q^{-z})^2$ and not $(1-q^{-z})^{-2}$). After some algebra, and using (2.25), we prove

$$M_n^{(2)} = \frac{\log^2 n}{\log^2 Q} + 2\frac{\log n}{\log Q}\left(\frac{\gamma}{\log Q} + \frac{1}{2}\right) + 2\beta - \frac{\gamma}{\log Q} - \frac{1}{2} + F(\log_Q n)$$

where β is given by

$$\beta = \frac{1}{h_1^2}\left\{\frac{\pi^2}{12} + \frac{\gamma^2}{2} + \frac{3}{4}\frac{h_2^2}{h_1^2} - \frac{1}{3}\frac{h_3}{h_1} + \frac{\gamma h_2}{h_1}\right\}.$$

The periodic function $F(x)$ is given in Proposition (ii). Using this and (2.8), we immediately prove (2.9) in Proposition (ii). Finally, formula (2.12) in Proposition (iii), is proved in a similar manner. In this case, we consider only the leading factor, which corresponds to D_n in (2.26) with the denominator replaced by $(1 - q^{-z})^k$ for the k-th moment of $\mathbf{M}_n$.

3. Numerical Work

In the following table, we present the values of M_n and var($\mathbf{M}_n$) for selected values of n, $2 \leq n \leq 200$. Observe that the asymptotics are accurate even for values of n below 10.

Table. ($p = 0.25$)

n	M_n	Asymptotic	var($\mathbf{M}_n$)	Asymptotic
2	5.71429	4.91586	15.18367	19.95905
10	10.68127	10.51036	18.80917	19.95905
20	13.00596	12.91978	19.36976	19.95905
30	14.38682	14.32920	19.56279	19.95905
40	15.37247	15.32920	19.66069	19.95905
50	16.13950	16.10486	19.71991	19.95905
60	16.76754	16.73862	19.75891	19.95905
70	17.29925	17.27446	19.78731	19.95905
80	17.76031	17.73862	19.80886	19.95905
90	18.16732	18.14804	19.82550	19.95905
100	18.53162	18.51428	19.83911	19.95905
110	18.86134	18.84559	19.85007	19.95905
120	19.16258	19.14804	19.85750	19.95905
130	19.43969	19.42628	19.86544	19.95905
140	19.69635	19.68388	19.87180	19.95905
150	19.93533	19.92370	19.87783	19.95905
160	20.15891	20.14804	19.88352	19.95905
170	20.36901	20.35878	19.88802	19.95905
180	20.56712	20.55746	19.89211	19.95905
190	20.75451	20.74540	19.89639	19.95905
200	20.93234	20.92370	19.89980	19.95905

The absolute error between the asymptotic value of M_n and the value given by the recurrence is 0.17 for $n = 10$. When $n = 100$, the error drops to 0.017, which is a tenth of the original error. For $n = 200$, the error is 0.008. In this example, the error appears to decrease roughly geometrically, halving for every increase of n by a hundred. In our proposition we have proved that the error is not larger than $O(n^{-1})$ (see also Remark (ii)). Clearly, when n takes on large values, the recurrence becomes impractical, both due to the amount of time required to compute it, as well as numerical overflow caused by the computation of factorial terms. The asymptotics give accurate values.

References

[1] Abramowitz, M. and Stegun, I., Handbook of Mathematical Functions, Dover, New York 1964.
[2] David, H., Order Statistics. Second Edition, John Wiley & Sons, New York 1981.

[3] Galambos, S., The Asymptotic Theory of Extreme Order Statistics. John Wiley & Sons, New York 1978.
[4] Henrici, P., Applied and Computational Complex Analysis, Vol. 2, John Wiley & Sons, New York, 1975.
[5] Knuth, D., The Art of Computer Programming. Sorting and Searching, Addison-Wesley, 1973.
[6] Rego, V., Mathur, A. P., Exploiting parallelism across program execution: A unification technique and its analysis, Purdue University, CSD TR-751, March 1988, to appear in IEEE Transactions on Computers, 1990.
[7] Szpankowski, W., On a recurrence equation arising in the analysis of conflict resolution algorithms, Stochastic Models, *3*, 89–114, 1987.
[8] Szpankowski, W., Some results on V-ary asymmetric tries, Journal of Algorithms, *9*, 224–244, 1988.
[9] Szpankowski, W., The evaluation of an alternative sum with applications to the analysis of some data structures, Information Processing Letters, *28*, 13–19, 1988.

W. Szpankowski
V. Rego
Department of Computer Sciences
Purdue University
West Lafayette, IN 47906
U.S.A.

Verleger: Springer-Verlag KG, Mölkerbastei 5, A-1010 Wien. — Herausgeber: Prof. Dr. Hans J. Stetter, Institut für Angewandte und Numerische Mathematik der Technischen Universität Wien, Wiedner Hauptstraße 6—10, A-1040 Wien. — Redaktion: Wiedner Hauptstraße 6—10, A-1040 Wien. — Satz und Umbruch: Asco Trade Typesetting Limited, Hong Kong; Reproduktion und Offsetdruck: Novographic, Ing. W. Schmid, Maurer-Lange-Gasse 64, A-1238 Wien. — Verlagsort: Wien. — Herstellungsort: Wien. — Printed in Austria.